AN ABSTRACT MORIMOTO THEOREM FOR GENERALIZED F-STRUCTURES

MARCO ALDI AND DANIELE GRANDINI

ABSTRACT. We abstract Morimoto's construction of complex structures on product manifolds to pairs of certain generalized F-structures on manifolds that are not necessarily global products. As an application we characterize invariant generalized complex structures on products in which one factor is a Lie group and generalize a theorem of Blair, Ludden and Yano on Hermitian bicontact manifolds.

1. INTRODUCTION

The study of generalized geometry in arbitrary (not necessarily even) dimension was pioneered by Vaisman [17] and further developed by various authors ([15],[16],[6],[1],[7]). The key notion is that of *generalized F-structure* i.e. a skew-symmetric endomorphism $\Phi : \mathbb{T}M \to \mathbb{T}M$ of the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$ of a manifold M, such that $\Phi^3 + \Phi = 0$. It easy to see that if Φ is a generalized F-structure on M, then the restriction of the tautological inner product to the kernel of Φ is nondegenerate on each fiber. In this paper we focus on a specific kind of generalized F-structures, for which $\ker(\Phi)$ has fiberwise split signature. Most natural examples of generalized F-structures, including generalized almost complex structures and generalized almost contact structures, have split signature. To study generalized F-structures, we find it convenient to first introduce the notion of *split structure* i.e. a subbundle $E \subseteq \mathbb{T}M$ on which the tautological inner product is nondegenerate and has split signature. A *split generalized F-structure* (or SGF-structure) is then defined to be an orthogonal, skew-symmetric endomorphism J of a split structure E.

The generalized tangent bundle is acted upon by the group $\mathrm{Diff}(M) \ltimes \Omega_{cl}(M)$ of extended diffeomorphism with closed forms acting by the so-called B-field transform. Infinitesimally, this action corresponds to the notion of *generalized Lie derivative* $\mathbb{L}_{\mathbf{x}}$ [9]. Given a subset $S \subseteq \Gamma(\mathbb{T}M \otimes \mathbb{C})$, it is useful to consider its *normalizer* $\mathbb{I}(S)$ i.e. the set of all sections $\mathbf{x}$ of $\mathbb{T}M \otimes \mathbb{C}$ such that $\mathbb{L}_{\mathbf{x}}(S) \subseteq S$. By definition, the normalizer of a split generalized F-structure $J \in \mathrm{End}(E)$ is the normalizer $\mathbb{I}(J)$ of its $\sqrt{-1}$-eigenbundle L_J. Geometrically, $\mathbb{I}(J)$ can be thought of the set of infinitesimal symmetries of

E that commute with J. According to [17], J is a *generalized* CRF-*structure* if $L_J \subseteq \mathbb{I}(J)$.

Given two SGF-structures J_1, J_2 it is natural to ask under what conditions L_{J_1} normalizes L_{J_2}. For instance, if $(M_1, \mathcal{J}_1)$ and $(M_2, \mathcal{J}_2)$ are generalized almost complex structures, then $\mathcal{J}_1$ (resp. $\mathcal{J}_2$) lifts to an SGF-structure J_1 (resp. J_2) on the split structure on $M_1 \times M_2$ generated by sections of $\mathbb{T}M_1$ (resp. of $\mathbb{T}M_2$). It is then easy to see that L_{J_1} and L_{J_2} normalize each other and that $J_1 \oplus J_2$ is a generalized complex structure if and only if both J_1 and J_2 are. Similarly, if $\mathcal{J}_1$ and $\mathcal{J}_2$ are SGF-structures corresponding to generalized almost contact structures on M_1 and M_2, one can still define their lifts J_1, J_2 to $M_1 \times M_2$, but $J_1 \oplus J_2$ is no longer a generalized almost complex structure for dimensional reasons. However, generalizing a classical construction of Morimoto, one can introduce a third SGF-structure Ψ on $M_1 \times M_2$ in such a way that $J_1 \oplus J_2 \oplus \Psi$ is a generalized almost complex structure. Extending a theorem of Morimoto [12] to the generalized setting, Gomez and Talvacchia [6] proved the existence of a canonical SGF-structure Ψ for which $J_1 \oplus J_2 \oplus \Psi$ is a generalized complex structure if and only if J_1, J_2 and Ψ are generalized CRF-structures and the natural framing of $L_\Psi \oplus \overline{L}_\Psi$ normalizes both J_1 and J_2.

In this paper we abstract the features that make Morimoto's construction [12] work into the concept of *(adaptable) Morimoto datum* defined out of: 1) mutually orthogonal split structures E_1, E_2, E_1' and E_2'; 2) SGF-structures J_1 on E_1, J_2 on E_2, Ψ on $E_1' \oplus E_2'$ and 3) global framings V_1 for E_1' and V_2 for E_2'. Our main results is an *Abstract Morimoto Theorem* stating that in presence of an adaptable Morimoto datum, $J_1 \oplus J_2 \oplus \Psi$ is a generalized CRF-structure if and only if (J_1, V_1) and (J_2, V_2) are *normal pairs*, an abstraction of the concept of normal generalized almost contact structure introduced in [1].

Our Abstract Morimoto Theorem unifies and extends several theorems à la Morimoto in the literature. If M is indeed a product $M_1 \times M_2$ and E_i, E_i' are pull-back of split structures on $\mathbb{T}M_i$, then our construction yields generalized almost complex structures on $M_1 \times M_2$ which simultaneously generalize Morimoto products of generalized almost contact structures [6] and Morimoto products of classical framed F-structures [13]. The generalized complex structures constructed with our method come in families and thus, even in the generalized contact case, they are more general than those of [6]. For instance, we show that the Morimoto product of two copies of the normal generalized almost contact structures on S^3 introduced in [1] yields holomorphic Poisson deformations of the Calabi-Eckmann complex structures on $S^3 \times S^3$ for every choice of complex structure on the T^2 fiber.

In a different direction, we are able to extend Sekiya's characterization of invariant generalized (almost) complex structures ([16], [1]) from products of the form $M \times \mathbb{R}$ to products of M with an arbitrary finite dimensional Lie group.

An important feature of the notion of Morimoto datum is that it is sufficiently flexible to apply to manifolds that are not necessarily global products. For instance, we are able to describe two constructions of generalized CRF-structures on flat principal bundles, one of which extends previous work [2] on normal contact pairs. A second class of examples of Morimoto data beyond the global product case comes from a generalized version of a classical theorem of Blair, Ludden and Yano [3] which states that Hermitian bicontact manifolds with bicontact forms (η_1, η_2) of bidegree $(1,1)$ are locally the product of normal contact manifolds. In this paper we prove an Abstract Blair-Ludden-Yano Theorem at the level of Hermitian bicontact data, a notion that we introduce in order to isolate the features of classical Hermitian bicontact structures of bidegree $(1,1)$ that we need. On the one hand, we prove that our Abstract Blair-Ludden-Yano Theorem implies the classical one. On the other hand, we show that this generalization is non-trivial since the non-commutative Calabi-Eckmann structures on $S^3 \times S^3$ provide non-classical examples of Hermitian bicontact data.

The paper is organized as follows. Section 2 is a recollection of basic notions and notations used in generalized geometry. We refer the reader to [8] and [9] for a systematic treatment of the subject. In Section 3 we define our main objects of study: split structures, SGF-structures and split generalized CRF-structures. In Section 4 we study normalizers of SGF-structures and introduce the important notion of normal pair. Section 5 contains the definition of Morimoto datum and the Abstract Morimoto Theorem. Section 6 is technical in nature and describes the behavior of normalizers and normal pairs under pull-back by a surjective submersion. In Section 7, Section 8 and Section 9 we specialize the Abstract Morimoto Theorem to various particular cases including global products and flat principal bundles, making the connection with previous results in the literature. We conclude with Section 10 in which we introduce the concept of Hermitian bicontact datum and prove the Abstract Blair-Ludden-Yano Theorem. In this paper, the notion of contact and bicontact datum is developed mainly for the purpose of providing non-trivial examples of Morimoto data. A systematic treatment of (bi)contact data, in particular exploring their connection with other attempts to extends contact geometry to the generalized setting (e.g. [15], [10]), would be interesting and we hope to come back to this point in the future.

2. Preliminaries on Generalized Geometry

Definition 1. The *generalized tangent bundle* of a real smooth manifold M of finite dimension n is the vector bundle $\mathbb{T}M := TM \oplus T^*M$. $\mathbb{T}M$ is endowed with a $C^\infty(M)$-bilinear, symmetric *tautological inner product* of signature (n, n) defined by

$$\langle X + \alpha, Y + \beta \rangle := \frac{1}{2}(\alpha(Y) + \beta(X))$$

for all $X, Y \in \Gamma(TM)$ and all $\alpha, \beta \in \Gamma(T^*M)$. The generalized tangent bundle is also endowed with an $\mathbb{R}$-bilinear map $[\ ,\ ] : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ called the *Dorfman bracket*, defined by

$$[X + \alpha, Y + \beta] := [X, Y] + \mathcal{L}_X\beta - \iota_Y d\alpha$$

for all $X, Y \in \Gamma(TM)$ and for all $\alpha, \beta \in \Gamma(T^*M)$. Sections of $\mathbb{T}M$ are denoted by $\mathbf{x}, \mathbf{y}$, etc. unless their (co)tangent components need to be specified.

Definition 2. For each $\mathbf{x} \in \Gamma(\mathbb{T}M)$, the *generalized Lie derivative with respect to* $\mathbf{x}$ is the $\mathbb{R}$-linear endomorphism $\mathbb{L}_{\mathbf{x}}$ of $\Gamma(\mathbb{T}M)$ defined by $\mathbb{L}_{\mathbf{x}}(\mathbf{y}) = [\mathbf{x}, \mathbf{y}]$ for all $\mathbf{y} \in \Gamma(\mathbb{T}M)$. $\mathbb{L}_{\mathbf{x}}$ extends to the unique endomorphism of the full tensor algebra of $\Gamma(\mathbb{T}M)$ such that $\mathbb{L}_{\mathbf{x}}(f) = 2\langle \mathbf{x}, df \rangle$ for all $f \in C^\infty(M)$ and such that $\mathbb{L}_x$ is a graded derivation with respect to the tensor product.

Remark 3. Let a be the projection of $\mathbb{T}M$ onto the tangent bundle TM. The quadruple $(\mathbb{T}M, \langle\ ,\ \rangle, [\ ,\ ], a)$ satisfies the axioms of *Courant algebroid*

i) $a(\mathbf{x})\,(\langle \mathbf{y}, \mathbf{z} \rangle) = \langle [\mathbf{x}, \mathbf{y}], \mathbf{z} \rangle + \langle \mathbf{y}, [\mathbf{x}, \mathbf{z}] \rangle$,
ii) $[\mathbf{x}, [\mathbf{y}, \mathbf{z}]] = [[\mathbf{x}, \mathbf{y}], \mathbf{z}] + [\mathbf{y}, [\mathbf{x}, \mathbf{z}]]$,
iii) $[\mathbf{x}, \mathbf{y}] + [\mathbf{y}, \mathbf{x}] = 2d\langle \mathbf{x}, \mathbf{y} \rangle$,

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \Gamma(\mathbb{T}M)$. These properties can be restated in terms of generalized Lie derivatives as follows

$$\mathbb{L}_{\mathbf{x}}\langle \mathbf{y}, \mathbf{z} \rangle = \langle \mathbb{L}_{\mathbf{x}}(\mathbf{y}), \mathbf{x} \rangle + \langle \mathbf{y}, \mathbb{L}_{\mathbf{x}}(\mathbf{z}) \rangle\,; \tag{1}$$

$$\mathbb{L}_{\mathbf{x}}[\mathbf{y}, \mathbf{z}] = [\mathbb{L}_{\mathbf{x}}(\mathbf{y}), \mathbf{z}] + [\mathbf{y}, \mathbb{L}_{\mathbf{x}}(\mathbf{z})]\,; \tag{2}$$

$$2d\langle \mathbf{x}, \mathbf{y} \rangle = \mathbb{L}_{\mathbf{x}}(\mathbf{y}) + \mathbb{L}_{\mathbf{y}}(\mathbf{x})\,; \tag{3}$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \Gamma(\mathbb{T}M)$.

Remark 4. It is well-known that given a closed three-form H on M, one may twist the Dorfman bracket to

$$[\mathbf{x}, \mathbf{y}]_H := [\mathbf{x}, \mathbf{y}] - \iota_{\mathbf{x}}\iota_{\mathbf{y}} H$$

which also satisfies the axioms of Courant algebroid. While the results of this paper rely only on these and therefore extend to the twisted case, we set $H = 0$ for notational convenience.

Notation 5. Given a subset S of $\Gamma(\mathbb{T}M)$, we denote the $C^\infty(M)$-submodule of $\Gamma(\mathbb{T}M)$ generated by S by $\operatorname{span}(S)$. We reserve the notation $\operatorname{span}_{\mathbb{R}}(S)$ for the $\mathbb{R}$-submodule of $\Gamma(M)$ generated by S.

Definition 6. Let E be a subbundle of $\Gamma(\mathbb{T}M)$. A *framing* of E is a real subspace V of $\Gamma(E)$ whose dimension equals the rank of E and such that $\operatorname{span}(V) = \Gamma(E)$. Moreover, if U is an open set in M, a *local framing* of E on U is a framing of $E_{|U}$.

3. Split structures

Definition 7. Let M be an n-dimensional manifold. A *split structure* on M of rank $2k$ is a subbundle $E \subseteq \mathbb{T}M$ such that the restriction $\langle , \rangle_{|E}$ is nondegenerate with signature (k, k). We denote by $\mathbb{E}_k(M)$ the set of all split structures of rank $2k$ on M, and we write $\mathbb{E}(M)$ for the set of all split structures on M.

Remark 8. Split structures are closed with respect to the following operations.

(1) If $E \in \mathbb{E}_k(M)$, then
$$E^\perp = \{\mathbf{x} \in \Gamma(\mathbb{T}M) \mid \langle \mathbf{x}, \mathbf{y}\rangle = 0 \,\text{for all}\; \mathbf{y} \in E\}$$
is a split structure of rank $2n - 2k$.

(2) Let $E \in \mathbb{E}(M)$, let $F : E \to \mathbb{T}M$ be a base preserving morphism and let C be a nowhere vanishing function on M such that
$$\langle F\mathbf{x}, F\mathbf{y}\rangle = C\langle \mathbf{x}, \mathbf{y}\rangle$$
for all $\mathbf{x}, \mathbf{y} \in \Gamma(E)$. Then $F(E) \in \mathbb{E}(M)$.

(3) If $E \in \mathbb{E}_k(M)$ and $E' \in \mathbb{E}_{k'}(M)$ are such that $\langle \Gamma(E), \Gamma(E')\rangle = 0$, then the Whitney sum $E \oplus E'$ is in $\mathbb{E}_{k+k'}(M)$.

(4) If $E \in \mathbb{E}_k(M)$ and $E' \in \mathbb{E}_{k'}(M')$ then the external Whitney sum $E \boxplus E'$ is in $\mathbb{E}_{k+k'}(M \times M')$. Note that the space of sections $\Gamma(E)$ (resp. $\Gamma(E')$) is included canonically into the space $\Gamma(E \boxplus E')$ as a $C^\infty(M)$-submodule (resp. $C^\infty(M')$-submodule), but not a $C^\infty(M \times M')$-submodule.

Remark 9. If $E \in \mathbb{E}_k(M)$ is equipped with a framing V,then the restriction of the tautological inner product to V is nondegenerate with signature (k, k). Moreover, the orthogonal group $\mathrm{O}(V) \subseteq O(E)$ can be identified (as a Lie group) with the subgroup endomorphisms Ψ such that $\Psi(V) \subseteq V$.

Definition 10. Let $E \in \mathbb{E}(M)$. A *split generalized F-structure on E* is a bundle endomorphism $J \in \operatorname{End}(E)$ which is skew-symmetric and orthogonal

with respect to the tautological inner product. We denote by $\mathrm{SGF}(E)$ the set of all almost complex split structures on E.

Remark 11. Split generalized F-structures are a particular case of the generalized F-structures introduced in [17]. In particular, the following two characterizations of $\mathrm{SGF}(E)$ can be easily deduced from the results of [17]. Extending $J \in \mathrm{SGF}(E)$ to $\mathbb{T}M$ by 0 provides a bijection between $\mathrm{SGF}(E)$ and the set of all orthogonal endomorphisms Φ of $\mathbb{T}M$ such that $\Phi^3 + \Phi = 0$ and $\ker(\Phi) = E$. On the other hand, assigning to J the subbundle

$$L_J = \{\mathbf{x} - \sqrt{-1}J\mathbf{x} \,|\, \mathbf{x} \in E\}$$

defines a bijection between $\mathrm{SGF}(E)$ and the set of maximally isotropic subbundles L of $E \otimes \mathbb{C}$ such that $L \cap \overline{L} = 0$.

Example 12. Viewing $\mathbb{T}M$ as split structure on M, $\mathrm{SGF}(\mathbb{T}M)$ coincides with the set of all generalized almost complex structures on M, as defined in [8].

Example 13. In [1], a *generalized almost contact structure* is defined as a pair (E, L) where $E \in \mathbb{E}_1(M)$ is a trivial subbundle of $\mathbb{T}M$ and L is a maximal isotropic subbundle of $E^\perp \otimes \mathbb{C}$ such that $L \cap \overline{L} = 0$. By Remark 11, for each trivial $E \in \mathbb{E}_1(M)$ there is a canonical bijection between $\mathrm{SGF}(E^\perp)$ and the set of generalized almost contact structures of the form (E, L). Let J be the split generalized F-structure on $E^\perp$ corresponding to a generalized almost contact structure (E, L) and let Φ be the extension of J to $\mathbb{T}M$ by 0. Given an isotropic frame $\{\mathbf{e}_1, \mathbf{e}_2\}$ of E such that $2\langle \mathbf{e}_1, \mathbf{e}_2\rangle = 1$ then $(\Phi, \mathbf{e}_1, \mathbf{e}_2)$ is a *generalized almost contact triple* as defined in [1]. Therefore, the set of generalized contact triples up to a change of frame of E can be identified with the union of all $\mathrm{SGF}(E^\perp)$, as E ranges over all rank 2 split structures on M that are trivial subbundles of $\mathbb{T}M$.

Example 14. If Φ is a classical F-structure in the sense of [17], then $\ker(\Phi) \cap TM$ and $\ker(\Phi) \cap T^*M$ are maximally isotropic in $\ker(\Phi)$. Therefore, the restriction of Φ to the orthogonal complement of $\ker(\Phi)$ is a split generalized F-structure.

Definition 15. A split generalized F-structure $J \in \mathrm{SGF}(E)$ is a *split generalized* CRF*-structure on* $E \in \mathbb{E}(M)$ if its $\sqrt{-1}$-eigenbundle L_J is closed under the Dorfman bracket. We denote by $\mathrm{CRF}(E)$ the set of all split generalized CRF-structures on E.

Example 16. The set of all generalized complex structures on M coincides with $\mathrm{CRF}(\mathbb{T}M)$.

Example 17. The following family of generalized almost contact structures on $M = S^3$ found in [1] will serve as a recurring example to illustrate the scope of the methods introduced in the present paper. Let $\{X_1, X_2, X_3\}$ be a global frame of TS^3 with dual frame $\{\alpha_1, \alpha_2, \alpha_3\} \subseteq T^*S^3$ such that $[X_i, X_j] = 2\varepsilon_{ijk}X_k$ and $[X_i, \alpha_j] = 2\varepsilon_{ijk}\alpha_k$, where ε_{ijk} is the Levi-Civita symbol. Given $h = f_2 + \sqrt{-1}f_3 \in C^\infty(S^3, \mathbb{C})$, we deform $\alpha_1, \alpha_2, \alpha_3$ in the generalized sense to

$$\begin{aligned} \mathbf{x}_1 &= \alpha_1 + f_2X_2 + f_3X_3\,, \\ \mathbf{x}_2 &= \alpha_2 - f_2X_1\,, \\ \mathbf{x}_3 &= \alpha_3 - f_3X_1\,. \end{aligned}$$

This leads to an interesting decomposition of $\mathbb{T}S^3$ as orthogonal direct sum of the split structures $E = \operatorname{span}(X_2, X_3, \mathbf{x}_2, \mathbf{x}_3)$ and $E' = \operatorname{span}(X_1, \mathbf{x}_1)$. For any h, we also consider the split generalized F-structure $J \in \mathrm{SGF}(E)$ defined by $J(X_2) = X_3$ and $J(\mathbf{x}_2) = \mathbf{x}_3$. If $h = 0$, we recover the standard almost contact structure on S^3 written in coordinates for which X_1 is tangent to the fibers of the Hopf fibration. A direct calculation shows that $J \in \mathrm{CRF}(E)$ if and only if $\overline{\partial}(h) = 0$, where $\partial = X_2 - \sqrt{-1}X_3$.

4. Normalizers and Normal Pairs

Definition 18. Let S be a subset of $\Gamma(\mathbb{T}M \otimes \mathbb{C})$. We say that a section $\mathbf{x}$ of $\mathbb{T}M \otimes \mathbb{C}$ *normalizes* S if $\mathbb{L}_{\mathbf{x}}(S) \subseteq S$. The set $\mathbb{I}(S)$ of all sections that normalize S is called the *normalizer* of S. If $T \subseteq \mathbb{T}M \otimes \mathbb{C}$ is a subbundle, we simply write $\mathbb{I}(T)$ for $\mathbb{I}(\Gamma(T))$.

Remark 19. Let $E \in \mathbb{E}(M)$. Given $\mathbf{x} \in \mathbb{I}(E)$, $\mathbf{y} \in \Gamma(E^\perp)$ and $\mathbf{z} \in \Gamma(E)$,

$$0 = \mathbb{L}_{\mathbf{x}}\langle \mathbf{y}, \mathbf{z}\rangle = \langle \mathbb{L}_{\mathbf{x}}(\mathbf{y}), \mathbf{z}\rangle + \langle \mathbf{y}, \mathbb{L}_{\mathbf{x}}(\mathbf{z})\rangle = \langle \mathbb{L}_{\mathbf{x}}(\mathbf{y}), \mathbf{z}\rangle$$

from which we conclude that $\mathbb{I}(E) = \mathbb{I}(E^\perp)$.

Definition 20. If J is a split generalized F-structure and L_J is its $\sqrt{-1}$-eigenbundle, we define the *normalizer of* J to be $\mathbb{I}(J) = \mathbb{I}(L_J)$. Given two split generalized F structures J_1 and J_2 we say that J_1 *normalizes* J_2 if $\Gamma(L_{J_1}) \subseteq \mathbb{I}(J_2)$.

Example 21. Let J be a split generalized F-structure on $E \in \mathbb{E}(M)$. Then $J \in \mathrm{CRF}(E)$ if and only if $\Gamma(L_J) \subseteq \mathbb{I}(J)$.

Remark 22. Let $E \in \mathbb{E}(M)$ and $J \in \mathrm{SGF}(E)$. Then $\mathbf{x} \in \mathbb{I}(J)$ if and only if $\mathbf{x} \in \mathbb{I}(E)$ and $\mathbb{L}_{\mathbf{x}}$ commutes with J as elements of $\mathrm{End}_{\mathbb{R}}(\Gamma(E))$. By extending the action of $\mathbb{L}_{\mathbf{x}}$ to $\mathrm{End}_{\mathbb{R}}(\Gamma(E))$, this last requirement can we rewritten as $\mathbb{L}_{\mathbf{x}}(J) = 0$.

Example 23. Consider a generalized almost contact triple $(\Phi, \mathbf{e}_1, \mathbf{e}_2)$ as in Example 13, let $E = \ker(\Phi)$ and let J be the restriction of Φ to $E^{\perp}$. In the language of [1], if $(\Phi, \mathbf{e}_1, \mathbf{e}_2)$ is integrable (resp. strongly integrable) then $J \in \mathrm{SGF}(E)$ is normalized by at least one of (resp. both) $\mathbf{e}_1$ and $\mathbf{e}_2$.

Lemma 24. *Let J be a split generalized* CRF*-structure on $E \in \mathbb{E}(M)$ and let $\mathbf{u} \in \mathbb{I}(J)$. Then $J(\mathbf{u}) \in \mathbb{I}(J)$.*

Proof: Let $\mathbf{v} = J(\mathbf{u})$. For every $\mathbf{x} \in \Gamma(E)$,

$$[\mathbf{u} - \sqrt{-1}\mathbf{v}, \mathbf{x} - \sqrt{-1}J(\mathbf{x})] = [\mathbf{u}, \mathbf{x}] - [\mathbf{v}, J(\mathbf{x})] - \sqrt{-1}([\mathbf{u}, J(\mathbf{x})] + [\mathbf{v}, \mathbf{x}])\,.$$

Since $J \in \mathrm{CRF}(E)$ and $\mathbf{u} \in \mathbb{I}(J)$, then

$$[\mathbf{u}, J(\mathbf{x})] + [\mathbf{v}, \mathbf{x}] = J([\mathbf{u}, \mathbf{x}] - [\mathbf{v}, J(\mathbf{x})]) = [\mathbf{u}, J(\mathbf{x})] - J[\mathbf{v}, J(\mathbf{x})]\,,$$

which in turn implies $\mathbf{v} \in \mathbb{I}(J)$. $\square$

Remark 25. Due to the local nature of the Dorfman bracket, the normalizer of a subbundle $S \subseteq \mathbb{T}M$ defines a sheaf on M, whose sections on an open set $U \subseteq M$ are given by

$$\mathbb{I}_U(S) := \{\mathbf{x} \in \Gamma_U(\mathbb{T}M) : \mathbb{L}_{\mathbf{x}}\Gamma_U(S) \subseteq \Gamma_U(S)\}\,.$$

Definition 26. A split structure $E \in \mathbb{E}(M)$ is said to be *complete* if $\Gamma(E)$ is locally generated by $\Gamma(E) \cap \mathbb{I}(E)$, i.e. if each $p \in M$ admits an open neighborhood U and a local framing W_U of E on U, such that $W_U \subseteq \Gamma_U(E) \cap \mathbb{I}_U(E)$.

Definition 27. Let $E, E' \in \mathbb{E}(M)$ be such that $E' \subseteq E^{\perp}$, let $J \in \mathrm{CRF}(E)$ and let V be a framing of E'. We say that (J, V) is a *normal pair* if $V \subseteq \mathbb{I}(J) \cap \mathbb{I}(E')$.

Example 28. If $J \in \mathrm{SGF}(\mathbb{T}M)$, then $(J, 0)$ is a normal pair if and only if J is a generalized complex structure.

Example 29. Let E, E' and J be as in Example 17 and consider the framing $V = \mathrm{span}_{\mathbb{R}}(X_1, \mathbf{x}_1)$ of E'. Then (J, V) is a normal pair if and only if h is annihilated by both $\overline{\partial}$ and $Y_1 = X_1 + 2\sqrt{-1}\mathrm{Id}$.

Example 30. More generally, let $(\Phi, \mathbf{e}_1, \mathbf{e}_2)$ be a generalized contact triple as in Example 13. Consider the framing $V = \mathrm{span}_{\mathbb{R}}(\mathbf{e}_1, \mathbf{e}_2)$ of $E = \ker \Phi$ and denote by J the restriction of Φ to E. Then (J, V) is a normal pair if and only if $(\Phi, \mathbf{e}_1, \mathbf{e}_2)$ is a normal generalized contact triple in the sense of [1]. In this case, the condition $V \subseteq \mathbb{I}(E)$ implies that the Dorfman bracket vanishes identically on V.

Lemma 31. *Let $E, E' \in \mathbb{E}(M)$ be such that $E' \subseteq E^{\perp}$. Given $J \in \mathrm{CRF}(E)$ and a framing V of E', the following are equivalent:*

i) (J,V) *is a normal pair;*
ii) $V \subseteq \mathbb{I}(J) \cap \mathbb{I}(E \oplus E')$*;*
iii) $V \subseteq \mathbb{I}(E) \cap \mathbb{I}(E \oplus E')$*;*
iv) $V \subseteq \mathbb{I}(E) \cap \mathbb{I}(E')$*.*

In particular, if $E' = E^{\perp}$*, then* (J,V) *is a normal pair if and only if* $V \subseteq \mathbb{I}(E)$*.*

Proof: $V \subseteq \mathbb{I}(J)$ is equivalent to $\mathbb{L}_{\mathbf{v}}(\Gamma(L)) \subseteq \Gamma(L)$ for all $\mathbf{v} \in V$, where L is the $\sqrt{-1}$-eigenspace of J. Since $V = \overline{V}$ and $E \otimes \mathbb{C} = L \oplus \overline{L}$, this implies $V \subseteq \mathbb{I}(E)$. Therefore, i)⇒ii)⇒iii). If iii) holds, then for every $\mathbf{v} \in V$, $\mathbf{e} \in \Gamma(E)$ and $\mathbf{e}' \in \Gamma(E^{\perp})$

$$\langle \mathbb{L}_{\mathbf{v}}\mathbf{e}', \mathbf{e}\rangle = \mathbb{L}_{\mathbf{v}}\langle \mathbf{e}', \mathbf{e}\rangle - \langle \mathbf{e}', \mathbb{L}_{\mathbf{v}}\mathbf{e}\rangle = 0$$

which in turns implies iv). Under the assumptions of iv), $\mathbb{L}_{\mathbf{v}}(\Gamma(L)) \subseteq \Gamma(E)$ for each $\mathbf{v} \in V$ and thus

$$\langle \mathbb{L}_{\mathbf{v}}\mathbf{x}, \mathbf{y}\rangle = -\langle \mathbb{L}_{\mathbf{x}}\mathbf{v}, \mathbf{y}\rangle = -\mathbb{L}_{\mathbf{x}}\langle \mathbf{v}, \mathbf{y}\rangle + \langle \mathbf{v}, \mathbb{L}_{\mathbf{x}}\mathbf{y}\rangle = 0$$

for every $\mathbf{x}, \mathbf{y} \in \Gamma(L)$. Therefore, $\mathbb{L}_{\mathbf{v}}(\Gamma(L)) \subseteq \Gamma(E \cap L^{\perp}) = \Gamma(L)$ for each $\mathbf{v} \in V$ and i) is proved. The last assertion follows from the equivalence of i) and iii). □

5. The Abstract Morimoto Theorem

Definition 32. Let $E'_1, E'_2 \in \mathbb{E}(M)$ such that $\langle E'_1, E'_2\rangle = 0$ and such that there exist framings $V_i \subseteq \Gamma(E'_i)$. Given $\Psi \in \mathrm{SGF}(E'_1 \oplus E'_2)$ we say that the triple (V_1, V_2, Ψ) is *admissible* if there exists an isomorphism $\phi : \Gamma(E'_1 \otimes \mathbb{C}) \to \Gamma(E'_2 \otimes \mathbb{C})$ of $C^{\infty}(M, \mathbb{C})$-modules such that

i) $\phi(V_1 \otimes \mathbb{C}) = V_2 \otimes \mathbb{C}$;
ii) $L_{\Psi} = \Gamma_{\phi}$;

where L_{Ψ} is the $\sqrt{-1}$-eigenbundle of Ψ and $\Gamma_{\phi} = \{\mathbf{e} + \phi(\mathbf{e}) \,|\, \mathbf{e} \in E'_1 \otimes \mathbb{C}\}$ is the graph of ϕ. If this is the case, we say that ϕ is an *admissible isomorphism* for the admissible triple (V_1, V_2, Ψ).

Example 33. Consider the product manifold $M = M_1 \times M_2$ in which each factor is a copy of S^3. For $i = 1, 2$ we pick global frames $\{X^i_1, X^i_2, X^i_3\}$ (resp. $\{\alpha^i_1, \alpha^i_2, \alpha^i_3\}$) of TM_i (resp. of T^*M_i and functions $h^i \in C^{\infty}(M_i, \mathbb{C})$ defining split structures $E_i, E'_i \in \mathbb{E}(M_i)$ and generalized F-structures $J_i \in \mathrm{SGF}(E_i)$, as in Example 17. Furthermore, let V_i be framings of E'_i as in Example 29. Fix $\tau = a + \sqrt{-1}b \in \mathbb{C} \setminus \mathbb{R}$ and let $\Psi \in \mathrm{SGF}(E'_1 \oplus E'_2)$ be such that $\Psi(X_1) = aX^1_1 + bX^2_1$ and $\Psi(\mathbf{x}^2_1) = b\mathbf{x}^1_1 - a\mathbf{x}^2_1$. If $\lambda = b/(a + \sqrt{-1})$, then ϕ defined by $\phi(X^1_1) = \lambda X^2_1$ and $\phi(\mathbf{x}^2_1) = -\lambda\mathbf{x}^1_1$ is an admissible isomorphism for the admissible triple (V_1, V_2, Ψ). If $h^1 = h^2 = 0$, then Ψ is the complex

structure of modulus τ on the elliptic fibers of the Calabi-Eckmann fibration $S^3 \times S^3 \to S^2 \times S^2$ described in [4].

Remark 34. Let $E_1', E_2' \in \mathbb{E}(M)$ be mutually orthogonal with global framings $V_i \subseteq \Gamma(E_i')$. Let $\Psi \in \mathrm{SGF}(E_1' \oplus E_2') \cap \mathrm{O}(V_1 \oplus V_2)$ be such that $\pi_{E_2'} \circ \Psi_{|V_1} : V_1 \to V_2$ is invertible. Here $\pi_{E_2'}$ denotes the orthogonal projection onto E_2'. Under these assumptions, (V_1, V_2, Ψ) is admissible. To see this, write

$$\Psi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with blocks corresponding to the decomposition $E_1' \oplus E_2'$. Admissibility implies that the maps B, C are invertible, and that

$$L_\Psi = \{\mathbf{e} - \sqrt{-1}A\mathbf{e} - \sqrt{-1}C\mathbf{e} : \mathbf{e} \in E_1' \otimes \mathbb{C}\} = \Gamma_\phi \,,$$

where $\phi = -B^{-1}(A - \sqrt{-1}\mathrm{Id})$. Note that in this case the admissible isomorphism ϕ is unique. Moreover, after a choice of orthonormal bases on V_1 and V_2 is made, the morphism Ψ is uniquely represented as a matrix

$$\Psi_0 = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \in \mathfrak{o}(2l, 2l) \cap \mathrm{O}(2l, 2l) \,,$$

where the admissibility translates into the condition $B_0, C_0 \in \mathrm{GL}(2l, \mathbb{R})$.In particular, the matrix

$$\Psi_0^{\mathrm{can}} = \begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}$$

yields the admissible triple used in the original work of Morimoto [12] and in some of its generalizations [13], [6], [7].

Proposition 35. *Let $E_1', E_2' \in \mathbb{E}_l(M)$ be mutually orthogonal with global framings $V_i \subseteq \Gamma(E_i')$. Let $\Sigma \subseteq \mathrm{SGF}(E_1' \oplus E_2') \cap \mathrm{O}(V_1 \oplus V_2)$ be the subset of all Ψ such that $\pi_{E_2'} \circ \Psi_{|V_1} : V_1 \to V_2$ is invertible. Then Σ is homeomorphic to $\mathrm{O}(l, l)$.*

Proof. The group $\mathrm{O}(V_1) \times \mathrm{O}(V_2)$ acts transitively on Σ by conjugation or, more precisely, by

$$(R_1, R_2) \cdot \Psi := R \Psi R^{-1} \,,$$

where $R = R_1 \oplus R_2 : V_1 \oplus V_2 \to V_1 \oplus V_2$. Given $\Psi_0 \in \Sigma$, Effros' Open Mapping Theorem [5] shows that the canonical bijection

$$\frac{\mathrm{O}(V_1) \times \mathrm{O}(V_2)}{\mathrm{Stab}(\Psi_0)} \to \Sigma$$

defined by $(R_1, R_2)\mathrm{Stab}(\Psi_0) \mapsto (R_1, R_2) \cdot \Psi_0$ is a homeomorphism. On the other hand, the stabilizer $\mathrm{Stab}(\Psi_0)$ consists of the pairs of the form

$(R_1, \phi_0 R_1 \phi_0^{-1})$ (where ϕ_0 is the admissible isomorphism of Ψ_0), and the projection $\mathrm{O}(V_1)\times\mathrm{O}(V_2)\to\mathrm{O}(V_2)$ descends to a homeomorphism

$$\frac{\mathrm{O}(V_1)\times\mathrm{O}(V_2)}{\mathrm{Stab}(\Psi_0)}\to\mathrm{O}(V_2)\,.$$

Combining these observations, we obtain the following chain of homeomorphisms

$$\Sigma\simeq\frac{\mathrm{O}(V_1)\times\mathrm{O}(V_2)}{\mathrm{Stab}(\Psi_0)}\simeq\mathrm{O}(V_2)\simeq\mathrm{O}(l,l)\,.$$

□

Remark 36. If $l=1$ then $O(1,1)$ is one dimensional and the construction of Remark 34 yields a one-parameter family of admissible triples. A particular instance is the τ-dependent family of admissible triples on $S^3\times S^3$ described in Example 33.

Lemma 37. *Let $E_1', E_2' \in \mathbb{E}(M)$ be such that $\langle E_1', E_2'\rangle = 0$ and let $V_i \subseteq \Gamma(E_i')\cap\mathbb{I}(E_i')$ be framings of E_i'. Given $\Psi\in\mathrm{SGF}(E_1'\oplus E_2')$ such that (V_1, V_2, Ψ) is an admissible triple, then $\Psi\in\mathrm{CRF}(E_1'\oplus E_2')$ if and only if $\phi([\mathbf{v},\mathbf{w}]) = [\phi(\mathbf{v}),\phi(\mathbf{w})]$ for all $\mathbf{v},\mathbf{w}\in V_1$.*

Proof: By assumption, $V_i\subseteq\mathbb{I}(E_i')$ and $\phi(V_1)\subseteq\mathbb{I}(E_2')$. Therefore, $[\mathbf{v},\phi(\mathbf{w})]\subseteq(E_1'\cap E_2')\otimes\mathbb{C}=0$ for any $\mathbf{v},\mathbf{w}\in V_1$. It follows that Ψ is integrable if and only if

$$0=\langle[\mathbf{v}+\phi(\mathbf{v}),\mathbf{w}+\phi(\mathbf{w})],\mathbf{z}+\phi(\mathbf{z})\rangle=\langle[\mathbf{v},\mathbf{w}],\mathbf{z}\rangle+\langle[\phi(\mathbf{v}),\phi(\mathbf{w})],\phi(\mathbf{z})\rangle$$

for every $\mathbf{v},\mathbf{w},\mathbf{z}\in V_1$. The isotropy of Γ_ϕ implies

$$\langle[\mathbf{v},\mathbf{w}],\mathbf{z}\rangle=-\langle\phi([\mathbf{v},\mathbf{w}]),\phi(\mathbf{z})\rangle\,,$$

which concludes the proof. □

Example 38. If Ψ is as in Remark 34 with $A=D=0$, then the admissible isomorphism ϕ maps V_1 to $\sqrt{-1}V_2$. If this is the case, Lemma 37 shows that $\Psi\in\mathrm{CRF}(E_1'\oplus E_2')$ if and only if the Dorfman bracket vanishes when restricted to V_1 and V_2.

Definition 39. Let $E_1, E_2, E_1', E_2'\in\mathbb{E}(M)$ be mutually orthogonal split structures. For $i=1,2$, denote $E_i''=E_i\oplus E_i'$ and let $E''=E_1''\oplus E_2''$. A *Morimoto datum* on M is given by $(J_1, J_2, V_1, V_2, \Psi)$, where $J_i\in\mathrm{SGF}(E_i)$, V_i is a framing of E_i' for $i=1,2$ and $\Psi\in\mathrm{SGF}(E_1'\oplus E_2')$, satisfies the following conditions:

1) $V_i\subseteq\mathbb{I}(E_1'')\cap\mathbb{I}(E_2'')$ for $i=1,2$;
2) there exist local framings $W_i\subseteq\mathbb{I}(E_1'')\cap\mathbb{I}(E_2'')$ of E_i for $i=1,2$;
3) (V_1, V_2, Ψ) is an admissible triple.

We say that a Morimoto datum is *adaptable* if the local framings W_i as above satisfy $d\langle J_i(W_i), W_i\rangle \subseteq \Gamma(E_i'')$. If such a W_i exists, we call it an *adapted local framing* of E_i.

Lemma 40. *Let $\mathcal{M} = (J_1, J_2, V_1, V_2, \Psi)$ be a Morimoto datum.*
i) If $J_i \in \mathrm{CRF}(E_i)$, then $\mathcal{M}$ is adaptable;
ii) If $\mathcal{M}$ is adaptable, then $[\Gamma(L_{J_1}), \Gamma(L_{J_2})] = 0$.

Proof: Let W_i be local framings of E_i as in Definition 39. Since $J_i \in \mathrm{CRF}(E_i)$, $[\mathbf{w} - \sqrt{-1}J_i(\mathbf{w}), \mathbf{z} - \sqrt{-1}J_i(\mathbf{z})]$ is in $\Gamma(E_i \otimes \mathbb{C})$ for each $\mathbf{w}, \mathbf{z} \in W_i$. Taking the imaginary part, $[J_i(\mathbf{w}), \mathbf{z}] + [\mathbf{w}, J_i(\mathbf{z})]$ is in $\Gamma(E_i)$. Since $2d\langle J_i(\mathbf{w}), \mathbf{z}\rangle = [J_i(\mathbf{w}), \mathbf{z}] + [\mathbf{z}, J_i(\mathbf{w})]$ and $[W_i, J_i(W_i)] \subseteq \Gamma(E_i'')$, we conclude that W_i is an adapted local framing and thus i) holds. Let W_1 and W_2 be respective adapted local framings of E_1 and E_2. Notice that $[W_1, W_2] \in E_1'' \cap E_2'' = 0$ and $[W_1, J_2(W_2)] \subseteq \Gamma(E_2'')$. On the other hand, for each $\mathbf{x} \in W_1$ and $\mathbf{y}, \mathbf{z} \in W_2$

$$0 = \mathbb{L}_{\mathbf{x}}\langle J_2(\mathbf{y}), \mathbf{z}\rangle = \langle \mathbb{L}_{\mathbf{x}}(J_2(\mathbf{y})), \mathbf{z}\rangle + \langle J_2(\mathbf{y}), \mathbb{L}_{\mathbf{x}}(\mathbf{z})\rangle = \langle \mathbb{L}_{\mathbf{x}}(J_2(\mathbf{y})), \mathbf{z}\rangle$$

which implies $[W_1, J_2(W_2)] = 0$. Similarly, $[J_1(W_1), W_2] = 0$ and therefore $[J_1(W_1), J_2(W_2)] \in \Gamma(E_1 \cap E_2) = 0$. In particular, for each $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$,

$$[\mathbf{w}_1 - \sqrt{-1}J_1(\mathbf{w}_1), \mathbf{w}_2 - \sqrt{-1}J_2(\mathbf{w}_2)] = 0\,.$$

This concludes the proof since each L_{J_i} is locally generated by sections of the form $\mathbf{w}_i - \sqrt{-1}J_i(\mathbf{w}_i)$. $\square$

Lemma 41. *Let $\mathcal{M} = (J_1, J_2, V_1, V_2, \Psi)$ be a Morimoto datum. Then (J_1, V_1) and (J_2, V_2) are both normal pairs if and only if*
1) $\mathcal{M}$ is adaptable;
2) J_1 and J_2 both normalize $J = J_1 \oplus J_2 \oplus \Psi$.

Proof: Let $\Gamma = \Gamma(L_J)$ and let $\Gamma_i = \Gamma(L_{J_i})$ for $i = 1, 2$. Since the normality of the pair (J_i, V_i) implies $J_i \in \mathrm{CRF}(E_i)$, Lemma 40 allows us to assume that $\mathcal{M}$ is adaptable and thus $[\Gamma_1, \Gamma_2] = 0$. Since $V_i \subseteq \mathbb{I}(E_i'')$, we see that $[\Gamma_i, \Gamma_\phi] = [\Gamma_i, V_i]$ which implies that $[\Gamma_i, \Gamma] \subseteq \Gamma$ if and only if

$$[\Gamma_i, \Gamma_i \oplus V_i] \subseteq \Gamma(L_J \cap E_i'') = \Gamma_i$$

if and only if (J_i, V_i) is a normal pair for $i = 1, 2$. $\square$

Theorem 42 (Abstract Morimoto Theorem)**.** *Let $\mathcal{M} = (J_1, J_2, V_1, V_2, \Psi)$ be a Morimoto datum. Then $\mathcal{M}$ satisfies*
i) $J = J_1 \oplus J_2 \oplus \Psi$ is a generalized CRF-structure;
ii) $\mathcal{M}$ is adaptable;
if and only if $\mathcal{M}$ satisfies

i') (J_1, V_1) *and* (J_2, V_2) *are normal pairs;*
ii') Ψ *is a generalized* CRF*-structure.*

Proof: If (J_1, V_1) and (J_2, V_2) are normal pairs and $\Psi \in \mathrm{CRF}(E_1' \oplus E_2')$, then $J \in \mathrm{CRF}(E'')$ and $\mathcal{M}$ is adaptable by Lemma 41. Conversely, if $J \in \mathrm{CRF}(E'')$ then in particular J_i normalizes $J_1 \oplus J_2 \oplus \Psi$. If in addition $\mathcal{M}$ is adaptable, then Lemma 41 implies that (J_1, V_1) and (J_2, V_2) are normal pairs. As a consequence of Lemma 31, $V_i \subseteq \mathbb{I}(E_i')$ for $i = 1, 2$. Therefore, the admissible triple (V_1, V_2, Ψ) satisfies the assumptions of Lemma 37 and therefore Ψ is a generalized CRF-structure. □

6. Flat Ehresmann connections

In this section we consider a surjective submersion $\pi : N \to M$ equipped with a flat Ehresmann connection, i.e. an involutive subbundle $H \subseteq TN$ such that

$$TN = H \oplus \ker(T\pi) .$$

The connection induces a splitting

$$\mathbb{T}N = (H \oplus \mathrm{Ann}(\ker(T\pi))) \oplus (\ker(T\pi) \oplus \mathrm{Ann}(H)) .$$

We refer to the split structures $H \oplus \mathrm{Ann}(\ker(T\pi))$ and $\ker(T\pi) \oplus \mathrm{Ann}(H)$, respectively, as the *horizontal* and *vertical split structure* defined by the connection H.

Remark 43. There is a canonical orthogonal isomorphism between

$$\pi^*\mathbb{T}M = \{(q, X_p + \alpha_p) : X_p + \alpha_p \in \mathbb{T}_pM, p \in M, q \in N, \pi(q) = p\}$$

and $H \oplus \mathrm{Ann}(\ker(T\pi))$ given by the map

$$(q, X_p + \alpha_p) \mapsto \hat{X}_q + \alpha_p \circ T_q\pi ,$$

where $\hat{X}_q \in H_q$ is uniquely defined by $(T_q\pi)(\hat{X}_q) = X_{\pi(q)}$. Under this identification, $\pi^*\mathbf{x} \in \Gamma(\pi^*\mathbb{T}M)$ is the *horizontal lifting* of $\mathbf{x} \in \Gamma(\mathbb{T}M)$. In particular, the restriction of π^* to $\Gamma(T^*M)$ coincides with the usual pull-back of forms.

Lemma 44. *For all* $\mathbf{x}, \mathbf{y} \in \Gamma(\mathbb{T}M)$, $[\pi^*\mathbf{x}, \pi^*\mathbf{y}] = \pi^*[\mathbf{x}, \mathbf{y}]$.

Proof: If $\mathbf{x}, \mathbf{y}$ are both forms, then both commutators vanish. If $\mathbf{x}, \mathbf{y}$ are both vector fields, the identity is a consequence of flatness. By linearity of the Dorfman bracket, it remains to consider the case $\mathbf{x} = \alpha \in \Gamma(T^*M)$ and $\mathbf{y} = X \in \Gamma(TM)$. Since

$$2\langle[\pi^*\alpha, \pi^*X], Y\rangle = (d\pi^*\alpha)(Y, \pi^*X) = (d\alpha)(T\pi Y, X) \circ \pi = 2\langle\pi^*[\alpha, X], Y\rangle$$

for all $Y \in \Gamma(TN)$, this shows that $[\pi^*\alpha, \pi^*X] = \pi^*[\alpha, X]$. Together with $d\langle\pi^*X, \pi^*\alpha\rangle = \pi^*d\langle X, \alpha\rangle$, this concludes the proof. □

Lemma 45. *If* $\mathbf{x} \in \Gamma(\mathbb{T}M)$, *then* $\pi^*\mathbf{x} \in \mathbb{I}(\ker(T\pi) \oplus \mathrm{Ann}(H))$.

Proof: If $\mathbf{v} \in \Gamma((\ker(T\pi) \oplus \mathrm{Ann}(H))$ and $\mathbf{x}, \mathbf{y} \in \Gamma(\mathbb{T}M)$, then

$$\langle [\pi^*\mathbf{x}, \mathbf{v}], \pi^*\mathbf{y}\rangle = -\langle \mathbf{v}, [\pi^*\mathbf{x}, \pi^*\mathbf{y}]\rangle = 0\,.$$

□

Proposition 46. *Let S be a (real or complex) subbundle of $\mathbb{T}M \otimes \mathbb{C}$ and let $\mathbf{x} \in \Gamma_U(E)$, for some open set $U \subseteq M$. Then, $\mathbf{x} \in \mathbb{I}_U(E)$ if and only if $\pi^*\mathbf{x} \in \mathbb{I}_{\pi^{-1}(U)}(\pi^*E)$.*

Proof: Let $\mathbf{x} \in \mathbb{I}_U(E)$, and let $U' \subseteq U$ be any open set that trivializes S. Given a frame $\{\mathbf{v}_i\}$ of E on U', then for all $\mathbf{w} \in \Gamma_{\pi^{-1}(U)}(\pi^*E)$, we have $\mathbf{w}_{\pi^{-1}(U')} = \sum_i f_i \pi^*\mathbf{v}_i$ for some smooth functions f_i defined on $\pi^{-1}(U')$, so that by Lemma 44

$$[\pi^*\mathbf{x}, \mathbf{w}]_{\pi^{-1}(U')} = \sum_i [\pi^*\mathbf{x}, f_i\pi^*\mathbf{v}_i]_{\pi^{-1}(U')} \in \Gamma_{\pi^{-1}(U')}(\pi^*(E))\,.$$

Here and below, $[-,-]_O$ denotes the restriction of the Dorfman bracket to an open set O. Since the open sets $\pi^{-1}(U')$ cover $\pi^{-1}(U)$, we obtain $[\pi^*\mathbf{x}, \mathbf{w}] \in \Gamma_{\pi^{-1}(U)}(\pi^*(E))$ and thus $\pi^*\mathbf{x} \in \mathbb{I}_{\pi^{-1}(U)}(\pi^*(E))$. Conversely, suppose that $\pi^*\mathbf{x} \in \mathbb{I}_{\pi^{-1}(U)}(\pi^*(E))$ and let $U' \subseteq U$, $\{\mathbf{v}_i\}$ be as before. If $\mathbf{z} \in \Gamma_U(E)$, then $[\pi^*\mathbf{x}, \pi^*\mathbf{z}] \in \Gamma_{\pi^{-1}(U)}(\pi^*E)$. On the other hand, $[\pi^*\mathbf{x}, \pi^*\mathbf{z}]_{\pi^{-1}(U')} = \sum_i g_i\pi^*\mathbf{v}_i$ and from Lemma 44 we obtain

$$\pi^*([\mathbf{x}, \mathbf{z}]_{U'}) = \sum g_i\pi^*\mathbf{v}_i\,.$$

It follows that $g_i = \pi^*h_i$, where h_i are smooth functions U' and

$$[\mathbf{x}, \mathbf{z}]_{U'} = \sum h_i\mathbf{v}_i \in \Gamma_{U'}(E).$$

Therefore, $[\mathbf{x}, \mathbf{z}] \in \Gamma_U(E)$ and the proof is complete. □

Corollary 47. *Let E, E' be orthogonal split structures on M, let $J \in \mathrm{SGF}(E)$ and let V be a framing of E'. Then (J, V) is a normal pair if and only if (π^*J, π^*V) is a normal pair.*

7. Morimoto products

For the remainder of the section we fix a product manifold $N = M_1 \times M_2$. In this case, we have submersions $\pi_i : N \to M_i$ given by the projections onto the two factors. As in Remark 43 we obtain flat connections $H_i := \ker(T\pi_j)$ and canonical isomorphisms

$$\pi_i^*(\mathbb{T}M_i) \cong H_i \oplus \mathrm{Ann}(H_j)$$

for $i \neq j$. Let us fix orthogonal split structures $E_i, E_i' \in \mathbb{E}(M_i)$, framings V_i of E_i' and split generalized F-structures J_i on E_i. We also define $E_i'' = E_i \oplus E_i'$, $E'' = E_1'' \oplus E_2''$ as well as

$$\tilde{E}_i := \pi_i^* E_i\,; \quad \tilde{E}_i' := \pi_i^* E_i'\,; \quad \tilde{J}_i := \pi_i^* J_i\,; \quad \tilde{V}_i := \pi_i^* V_i\,.$$

Note that $\tilde{E}_1, \tilde{E}_2, \tilde{E}_1', \tilde{E}_2'$ are mutually orthogonal split structures on N, $\tilde{V}_i$ is a framing of $\tilde{E}_i'$ and $\tilde{J}_i \in \mathrm{SGF}(\tilde{E}_i)$.

Definition 48. Let E_i, E_i', J_i, V_i as above. Then $(J_1, J_2, V_1, V_2, \Psi)$ is an *external Morimoto datum on* N is given by if $\Psi \in \mathrm{SGF}(E_1' \boxplus E_2')$ satisfies the following conditions:

1) $V_i \subseteq \mathbb{I}(E_i'')$ for $i = 1, 2$;
2) there exist local framings $W_i \subseteq \mathbb{I}(E_i'')$ of E_i for $i = 1, 2$;
3) $(\pi_1^* V_1, \pi_2^* V_2, \Psi)$ is an admissible triple.

An external Morimoto datum is called *adaptable* if the local framings W_i of condition 2) additionally satisfy $d\langle W_i, J_i W_i\rangle \subseteq \Gamma(E_i'')$ for $i = 1, 2$.

Remark 49. If $E_1' = E_1^\perp$ and $E_2' = E_2^\perp$, then conditions 1) and 2) in Definition 48 are trivially satisfied. Moreover, in this case all Morimoto data are adaptable.

Lemma 50. *If* $(J_1, J_2, V_1, V_2, \Psi)$ *is an (adaptable) external Morimoto datum, then* $(\pi_1^* J_1, \pi_2^* J_2, \pi_1^* V_1, \pi_2^* V_2, \Psi)$ *is an (adaptable) Morimoto datum.*

Proof: Let $(J_1, J_2, V_1, V_2, \Psi)$ be an external Morimoto datum, and let $\mathbf{v} \in V_i$. By Proposition 46 and Lemma 45, $V_i \subseteq \mathbb{I}(\pi_1 * E_1'') \cap \mathbb{I}(\pi_2^* E_2'')$. Similarly, if W_i is a local framing of E_i as in Definition 48, then $\pi_i^* W_i$ is a local framing of $\pi_i^* E_i$ such that $\pi_i^* W_i \subseteq \mathbb{I}(\pi_1^* E_1'') \cap \mathbb{I}(\pi_2^* E_2'')$. The adaptability of W_i implies $d\langle \pi_i^* J_i \pi_i^* W_i, \pi_i^* W_i\rangle = \pi_i^* d\langle J_i W_i, W_i\rangle \subseteq \Gamma(\pi_i^* E_i'')$, which concludes the proof. $\square$

Definition 51. Let $(J_1, J_2, V_1, V_2, \Psi)$ be an external Morimoto datum for $M_1 \times M_2$. We define the *Morimoto product of* J_1 *and* J_2 *with respect to* Ψ to be

$$J_1 \boxplus_\Psi J_2 := \pi_1^* J_1 \oplus \pi_2^* J_2 \oplus \Psi \in \mathrm{SGF}(E'')\,.$$

Theorem 52. *Let* $(J_1, J_2, V_1, V_2, \Psi)$ *be an adaptable external Morimoto datum for* $M_1 \times M_2$*. Then* $J_1 \boxplus_\Psi J_2 \in \mathrm{CRF}(E'')$ *if and only if*

i) (J_1, V_1) *and* (J_2, V_2) *are normal pairs;*
ii) $\Psi \in \mathrm{CRF}(E_1' \boxplus E_2')$.

Proof: The result is a direct consequence of the Abstract Morimoto Theorem, which can be applied because of Lemma 50 and Corollary 47. $\square$

Remark 53. If J_i are generalized almost complex structures, then $V_1 = V_2 = 0$ and thus $\Psi = 0$. In this case, Theorem 52 amounts to the assertion that $J_1 \boxplus J_2$ is integrable if and only if both J_1 and J_2 are integrable.

Corollary 54. *Let J_i be generalized almost contact structures, let $E_i' = E_i^\perp$ and let (V_1, V_2, Ψ) be admissible. Then $J_1 \boxplus_\Psi J_2$ is a generalized complex structure on M if and only if (J_1, V_1) and (J_2, V_2) are normal pairs, i.e. the generalized almost contact triples associated with (J_i, V_i) in Example 13 are normal.*

Proof: It suffices to observe that since dim $V_i = 2$, the normality of (J_i, V_i) implies that the Dorfman bracket vanishes identically on V_i. Therefore, the admissible isomorphism ϕ satisfies

$$[\phi(\mathbf{v}), \phi(\mathbf{w})] = 0 = \phi[\mathbf{v}, \mathbf{w}]$$

for all $\mathbf{v}, \mathbf{w} \in V_i$. $\square$

Remark 55. In particular, the integrability of Morimoto products of generalized almost contact structures does not depend on the choice of admissible triple.

Example 56. If $M_1 = M_2 = S^3$ and τ, J_i, V_i, Ψ are as in Example 33, then $(J_1, J_2, V_1, V_2, \Psi)$ is an adaptable external Morimoto datum on $M = M_1 \times M_2$. According to Corollary 54 and Example 29, $J = J_1 \boxplus_\Psi J_2$ is integrable if and only if $h^i \in \ker(\overline{\partial}^i) \cap \ker(Y_1^i)$ for $i = 1, 2$. If $h^1 = h^2 = 0$, these conditions are trivially satisfied and J coincides with the family (parametrized by τ) of complex structures on $S^3 \times S^3$ discovered in [4]. On the other hand if J is integrable and $(h^1, h^2) \neq 0$, then J is a generalized complex structure that preserves TM but not T^*M. As observed in [11], this implies that turning on the parameters h^i has the effect of deforming the complex structure of Calabi and Eckmann by means of a holomorphic Poisson bivector. Therefore, the Morimoto product of two of the normal generalized almost contact structures on S^3 described in [1] with respect to split generalized F-structures Ψ introduced in Example 33 is a (generically non-commutative) Calabi-Eckmann structure on $S^3 \times S^3$.

8. Products with Lie Groups

For the remainder of this section let us fix a finite-dimensional Lie group G with identity e and a manifold M. We denote by $\mathfrak{g}$ the Lie algebra of G we fix a basis $\{b_i\}$ of $\mathfrak{g} \ltimes \mathfrak{g}^* = \mathbb{T}_e G$. We also consider the left-action of G acts on $M \times G$ defined by $h(p, g) := (p, hg)$ for all $p \in M$ and $g, h \in G$.

Theorem 57. *The following sets are in canonical bijection*

i) G-invariant generalized almost complex structures $\mathcal{J}$ on $M \times G$, such that $\pi_{\mathbb{T}M}\mathcal{J}_{|\mathbb{T}G}$ is fiberwise injective, with image of split signature;
ii) quadruples $(E, J, \{\mathbf{v}_i\}, \varphi)$, where $E \in \mathbb{E}(M)$, $J \in \mathrm{SGF}(E)$, $\{\mathbf{v}_i\}$ is a global frame of $E^\perp$ and $\varphi : M \to \mathfrak{o}(\mathfrak{g} \ltimes \mathfrak{g}^*)$ is a smooth map such that, for all $p \in M$

$$\langle(\varphi_p^2 + \mathrm{Id}_{\mathfrak{g}\ltimes\mathfrak{g}^*})b_i, b_j\rangle = \langle \mathbf{v}_i(p), \mathbf{v}_j(p)\rangle\,.$$

Proof: Given $\mathcal{J}$, for all $(p, g) \in M \times G$ we have

$$\mathcal{J}_{p,g} = \begin{bmatrix} A_{p,g} & B_{p,g} \\ C_{p,g} & D_{p,g} \end{bmatrix},$$

with respect to the decomposition $\mathbb{T}_{(p,g)}(M \times G) = \mathbb{T}_pM \oplus \mathbb{T}_gG$. If $\mathbf{v}_i(p) := B_{p,e}(b_i)$, then $E' := \mathrm{span}(\{\mathbf{v}_i\}) \subseteq \mathbb{T}M$ is a split structure and so is $E := (E')^\perp$. Let J be defined by $J_p := A_{p,e}|_{E_p}$ for each $p \in M$. Since $E_p = \ker(C_{p,e})$, $J \in \mathrm{SGF}(E)$ and φ defined by $\varphi_p := D_{p,e}$ for each $p \in M$ is the required map. Conversely, consider a quadruple $(E, \{\mathbf{v}_i\}, J, \varphi)$. For each $p \in M$, define

$$\Psi_{p,e} : E_p^\perp \oplus \mathbb{T}_eG \to E_p^\perp \oplus \mathbb{T}_eG$$

such that

$$\Psi_{p,e} = \begin{bmatrix} -B_{p,e}\varphi_p B_{p,e}^{-1} & B_{p,e} \\ -B_{p,e}^* & \varphi_p \end{bmatrix},$$

where $B_{p,e} : \mathbb{T}_eG \to \mathbb{T}_pM$ is the isomorphism defined by $B_{p,e}(b_i) := \mathbf{v}_i(p)$. This map extends uniquely to a G-invariant bundle endomorphism $\Psi \in \mathrm{SGF}(E^\perp \boxplus \mathbb{T}G)$. Let π_1, π_2 be the projections of $M \times G$ onto the respective factors. If $V_1 = \mathrm{span}_{\mathbb{R}}(\{\mathbf{v}_i\})$ and V_2 is the space of left-invariant sections of $\mathbb{T}G$, then $(\pi_1^*V_1, \pi_2^*V_2, \Psi)$ is an admissible triple which gives rise to the Morimoto product

$$\mathcal{J} := J \boxplus_\Psi 0\,.$$

The assignments $\mathcal{J} \mapsto (E, \{\mathbf{v}_i\}, J, \varphi)$ and $(E, \{\mathbf{v}_i\}, J, \varphi) \mapsto \mathcal{J}$ just described provide the required canonical bijections. □

Example 58. If $G = \mathbb{R}$ the condition $\langle(\varphi_p^2 + \mathrm{Id})b_i, b_j\rangle = \langle \mathbf{v}_i(p), \mathbf{v}_j(p)\rangle$ and, correspondingly, the condition that $\pi_{\mathbb{T}M}\mathcal{J}(\mathbb{T}\mathbb{R})$ is of split signature are both automatically satisfied. Therefore, Theorem 57 reduces to Sekiya's characterization [16] of invariant generalized almost complex structure on $M \times \mathbb{R}$.

Corollary 59. *Let $\mathcal{J}$ be a G-invariant generalized almost complex structure on $M \times G$, such that $B := \pi_{\mathbb{T}M}\mathcal{J}_{|\mathbb{T}G}$ is fiberwise injective, with image of split signature. Let $(E, J, \{\mathbf{v}_i\}, \varphi)$ corresponding to $\mathcal{J}$ under the bijection*

of Theorem 57. Then, $\mathcal{J}$ is integrable if and only if $(J, \mathrm{span}_{\mathbb{R}}(\{\mathbf{v}_i\}))$ is a normal pair and the map

$$\phi = (B^*)^{-1} \circ (\varphi - \sqrt{-1}\mathrm{Id})$$

satisfies $[\phi(\mathbf{v}), \phi(\mathbf{w})] = \phi[\mathbf{v}, \mathbf{w}]$ for all $\mathbf{v}, \mathbf{w} \in \Gamma(\mathbb{T}G)$.

Proof: As in the proof of Theorem 57, $\mathcal{J}$ can be written as a Morimoto product of the form $J \boxplus_{\Psi} 0$. By Remark 49, the corresponding Morimoto datum is adaptable. Theorem 52 then guarantees that $\mathcal{J}$ is integrable if and only if $(J, \mathrm{span}_{\mathbb{R}}(\{\mathbf{v}_i\}))$ is a normal pair and Ψ is a split generalized CRF-structure. The result then follows from Lemma 37. □

Example 60. Let $G = \mathbb{R}^k$, and assume that $\mathcal{J}$ satisfies the conditions of Theorem 57. Then Example 38 shows that $\mathcal{J}$ is integrable if and only if $(J, \mathrm{span}_{\mathbb{R}}(\{\mathbf{v}_i\}))$ is a normal pair and $[\mathbf{v}_i, \mathbf{v}_j] = 0$ for all i, j.

Example 61. Let J be a split generalized F-structure defined by a classical F-structure on a manifold M as in Example 14. Suppose that J, together with vectors $\{v_i\} \subseteq \Gamma(TM)$, endows M with the structure of *f-manifold with complemented frame* in the sense of [13]. Let Ψ_0^{can} be as in Remark 34 with respect to the basis consisting of the complemented frame $\{\mathbf{v}_i\}$ extended by the standard orthogonal basis of invariant sections of $\mathbb{T}\mathbb{R}^k$. By definition, M is a *normal framed f-manifold* if the generalized almost complex structure $J \boxplus_{\Psi_0^{\mathrm{can}}} 0$ is integrable. By Example 60, we see that M is a normal framed f manifold if and only if $(J, \mathrm{span}_{\mathbb{R}}(\{\mathbf{v}_i\}))$ is a normal pair and $[\mathbf{v}_i, \mathbf{v}_j] = 0$ for all i, j. In [13], Nakagawa proved the following generalization of Morimoto's Theorem: the Morimoto product $J_1 \boxplus_{\Psi_0^{\mathrm{can}}} J_2$ of two framed f-manifolds $(M_1, J_1, \{\mathbf{v}_{1,i}\})$ and $(M_2, J_2, \{\mathbf{v}_{2,i}\})$ is integrable if and only if M_1 and M_2 are normal framed f-manifolds. Thanks to Example 38, one may view Nakagawa's Theorem as a particular case of Theorem 52.

9. Flat principal bundles

For the reminder of this section, let $\pi : N \to M$ be a principal bundle with fiber G admitting a flat connection H. As customary in this context, we assume H to be G-invariant, so that the vertical and horizontal split structures are G-invariant as well. If a basis $\{v_i\}$ of the Lie algebra of G is fixed and $\tilde{v}_i \in \Gamma(TN)$ denotes the fundamental vector field generated by v_i, then $\ker(T\pi)$ is trivialized by the global frame $\{\tilde{v}_i\}$ while $\mathrm{Ann}(H)$ is trivialized by the dual global coframe $\{\tilde{v}_i^*\}$. In particular, the vertical split structure $\ker(T\pi) \oplus \mathrm{Ann}(H)$ is a trivial bundle. Moreover, the framing $V' = \mathrm{span}_{\mathbb{R}}(\{\tilde{v}_i, \tilde{v}_j^*\}_{i,j})$ of the vertical split structure is involutive, i.e. it is closed under the Dorfman bracket.

Lemma 62. *Let E, E' be orthogonal split structures on M, let $J \in \mathrm{SGF}(E)$ and let V be a framing of E'. If $E'' = E \oplus E'$, then*

i) $V' \subseteq \mathbb{I}(\pi^* E'') \cap \mathbb{I}(\ker(T\pi) \oplus \mathrm{Ann}(H))$;
ii) $\pi^* V \subseteq \mathbb{I}(\pi^* E'') \cap \mathbb{I}(\ker(T\pi) \oplus \mathrm{Ann}(H))$ *if and only if* $V \subseteq \mathbb{I}(E'')$.

Proof: In order to prove the first statement, let $\mathbf{v}, \mathbf{w} \in V'$ and let $\mathbf{u} \in \Gamma(\mathbb{T}M)$. From the involutivity of V', it follows that $\langle [\mathbf{v}, \mathbf{w}], \pi^* \mathbf{u} \rangle = 0$ and thus V' normalizes the vertical split structure. Similarly, if $\mathbf{e} \in \Gamma(E'')$ then

$$\langle [\mathbf{v}, \pi^* \mathbf{e}], \mathbf{w} \rangle = -\langle \pi^* \mathbf{e}, [\mathbf{v}, \mathbf{w}] \rangle = 0 \, .$$

This, together with

$$\langle [\mathbf{v}, \pi^* \mathbf{e}], \pi^* \mathbf{u} \rangle = \langle \mathbf{v}, [\pi^* \mathbf{e}, \pi^* \mathbf{u}] \rangle = 0 \, ,$$

shows that $[\mathbf{v}, \pi^* \mathbf{e}] = 0$ and thus V' normalizes $\pi^* E''$. The second statement is a direct consequence of Lemma 45 and Proposition 46. $\square$

Theorem 63. *In addition to the assumptions of Lemma 62, suppose that*

i) $V \subseteq \mathbb{I}(E'')$;
ii) E' *admits local framings such that* $W \subseteq \mathbb{I}(E'')$ *and* $d\langle W, JW \rangle \subseteq \Gamma(E'')$;
iii) $(\pi^* V, V', \Psi)$ *is an admissible triple.*

Then $\pi^* J \oplus \Psi \in \mathrm{CRF}(\pi^* E'' \oplus \ker(T\pi) \oplus \mathrm{Ann}(H))$ *if and only if* (J, V) *is a normal pair and the admissible isomorphism* $\phi : V' \to \pi^* V$ *is a Lie algebra isomorphism.*

Proof: Our assumptions, together with Lemma 62 imply that $(\pi^* J, 0, \pi^* V, V', \Psi)$ is an adaptable Morimoto datum. The result then follows combining the Abstract Morimoto Theorem, Corollary 47 and Lemma 37. $\square$

Remark 64. Note that any flat principal G-bundle on M can be written in the form $N = (\widetilde{M} \times G)/\pi_1(M)$, where $\widetilde{M}$ is the universal cover of M and $\pi_1(M)$ acts on G by holonomy. This point of view suggests an alternative method to construct split generalized F-structures on N. Start from a structure on M, lift it to a $\pi_1(M)$-invariant structure on $\widetilde{M}$, take a Morimoto product with a $\pi_1(M)$-invariant structure on G and descend the resulting structure to N. In the context of classical contact geometry, this (in the more general context of flat bundles) is described in [2]. This should be contrasted with Theorem 63 in which G is not endowed with split generalized F structures and instead an admissible triple is used to extend the SGFstructure on $\pi^* \mathbb{T}M$ to a possible larger split structure.

10. Abstract Blair-Ludden-Yano Theorem

Definition 65. Let $E \in \mathbb{E}(M)$. We say that (V, W) is a *split framing of* E if V and W are isotropic and $V \oplus W$ is a framing of E.

Definition 66. Let $E \in \mathbb{E}(M)$ and $E' \in \mathbb{E}_k(M)$ be mutually orthogonal. Given a maximal isotropic subbundle $L \subseteq E$ and a split framing (V, W) for E', we say that (L, V, W) is a *rank k contact datum for* (E, E') if

1) $V \subseteq \mathbb{I}(E)$;
2) $\Gamma(L) \oplus W \subseteq \mathbb{I}(\Gamma(L) \oplus W)$;
3) $[V \oplus W, V \oplus W] = 0$;
4) $\Gamma(L) = \mathbb{L}_W(\Gamma(L))$;
5) $\Gamma(E) = \Gamma(L) \oplus \mathbb{L}_V(\Gamma(L))$.

Remark 67. If $E' = E^\perp$, then 3) implies $\langle [V \oplus W, \Gamma(E)], V \oplus W\rangle = 0$. In turn, this shows that condition 1) is automatically satisfied and that condition 2) simplifies to $\Gamma(L) \subseteq \mathbb{I}(\Gamma(L) \oplus W)$.

Remark 68. If $E' \in \mathbb{E}_1(M)$ a split framing (V, W) is uniquely determined by V. This observation allows us to use the shorthand notation (L, V) for a rank 1 contact datum (L, V, W).

Lemma 69. *If (L, V) is a rank 1 contact datum for (E, E'), then*

i) $\mathbb{L}_V(L) \subseteq E$ is maximal isotropic;
ii) $W \subseteq \mathbb{I}(E)$.

Proof: If $\mathbf{e}$ is a generator of V and $\mathbf{x}, \mathbf{y} \in \Gamma(L)$, then

$$\langle \mathbb{L}_{\mathbf{e}}\mathbf{x}, \mathbb{L}_{\mathbf{e}}\mathbf{y}\rangle = \langle \mathbb{L}_{\mathbf{x}}\mathbf{e}, \mathbb{L}_{\mathbf{y}}\mathbf{e}\rangle = \mathbb{L}_{\mathbf{y}}\langle \mathbb{L}_{\mathbf{x}}\mathbf{e}, \mathbf{e}\rangle - \langle \mathbf{e}, \mathbb{L}_{\mathbf{x}}\mathbb{L}_{\mathbf{y}}\mathbf{e}\rangle = -\langle \mathbf{e}, \mathbb{L}_{\mathbf{x}}\mathbb{L}_{\mathbf{y}}\mathbf{e}\rangle\,.$$

Similarly,

$$\langle \mathbb{L}_{\mathbf{e}}\mathbf{x}, \mathbb{L}_{\mathbf{e}}\mathbf{y}\rangle = -\langle \mathbb{L}_{\mathbf{y}}\mathbb{L}_{\mathbf{x}}\mathbf{e}, \mathbf{e}\rangle = \langle \mathbb{L}_{\mathbf{x}}\mathbb{L}_{\mathbf{y}}\mathbf{e}, \mathbf{e}\rangle - \langle \mathbb{L}_{[\mathbf{y},\mathbf{x}]}\mathbf{e}, \mathbf{e}\rangle = \langle \mathbb{L}_{\mathbf{x}}\mathbb{L}_{\mathbf{y}}\mathbf{e}, \mathbf{e}\rangle$$

from which i) follows. To prove ii) observe that for each $\mathbf{w} \in W$

$$\langle \mathbb{L}_{\mathbf{w}}\mathbf{x}, \mathbf{e}\rangle = \mathbb{L}_{\mathbf{w}}\langle \mathbf{x}, \mathbf{e}\rangle - \langle \mathbf{x}, \mathbb{L}_{\mathbf{w}}\mathbf{e}\rangle = 0$$

implies $W \subseteq \mathbb{I}(L)$. On the other hand,

$$\mathbb{L}_{\mathbf{w}}(\mathbb{L}_{\mathbf{e}}\mathbf{x}) = \mathbb{L}_{[\mathbf{w},\mathbf{e}]}\mathbf{x} - \mathbb{L}_{\mathbf{e}}(\mathbb{L}_{\mathbf{w}}\mathbf{x}) \in \mathbb{L}_V(L)$$

shows that $W \subseteq \mathbb{I}(\mathbb{L}_V(L))$ which concludes the proof. □

Example 70. Consider a contact form η on M and a corresponding Reeb vector field ξ. If $E' = \mathrm{span}(\xi, \eta)$ and $E = (E')^\perp$, then $(TM \cap E, \mathrm{span}(\eta))$ is a rank 1 contact datum.

Example 71. In the notation of Example 17, let $L = \mathrm{span}(X_2, X_3)$ and $V = \mathrm{span}_{\mathbb{R}}(\mathbf{x}_1)$. Thanks to Remark 67, (L, V) is a rank 1 contact datum for (E, E') if and only if

$$0 = [X_1, \mathbf{x}_1] = \mathrm{Re}(Y_1(h))X_2 + \mathrm{Im}(Y_1(h))X_3$$

and hence if and only if $Y_1(h) = 0$. If $h = 0$ this is a particular case of Example 70. On the other hand, if $h \neq 0$ then $\mathbf{x}_1$ is no longer a 1-form and therefore the resulting contact datum is not defined by a classical contact structure.

Definition 72. Let (L, V) be a rank 1 contact datum for (E, E') and let $J \in \mathrm{SGF}(E)$. We say that (J, L, V) is a *normal contact datum for* (E, E') if $J(L \oplus \mathrm{span}(W)) \subseteq L \oplus \mathrm{span}(W)$ and $(J, V \oplus W)$ is a normal pair.

Remark 73. If $E' = E^{\perp}$, then combining Lemma 69, Remark 67 and Lemma 31 we see that $(J, V \oplus W)$ is a normal pair if and only if $J \in \mathrm{CRF}(E)$.

Example 74. Let ξ and η be as in Example 70 and let $\phi \in \mathrm{End}(TM)$ be such that (ϕ, ξ, η) is a classical almost complex structure. If J denotes the split generalized F-structure induced by ϕ on E, then $(J, TM \cap E, \mathrm{span}_{\mathbb{R}}(\eta))$ is a normal contact datum if and only if (ϕ, ξ, η) is a normal almost contact structure.

Example 75. Let (L, V) be the rank 1 contact datum of Example 71 and let J be as in Example 17. Then (J, L, V) is a normal contact datum if and only if $h \in \ker(\overline{\partial}) \cap \ker(Y_1)$.

Definition 76. Let E, E'_1, E'_2 be mutually orthogonal split structures with E'_1 and E'_2 of rank 1. Given a maximal isotropic subbundle $L \subseteq E$ and split framings (V_1, W_1) for E_1 and (V_2, W_2) for E_2, we say that (L, V_1, V_2) is a *bicontact datum for* (E, E'_1, E'_2) if

1) $(L, V_1 \oplus V_2, W_1 \oplus W_2)$ is a rank 2 contact datum for $(E, E'_1 \oplus E'_2)$;
2) $L = L_1 \oplus L_2$, where $L_1 = \ker(\mathbb{L}_{V_2}) \cap L$ and $L_2 = \ker(\mathbb{L}_{V_1}) \cap L$;
3) For $i = 1, 2$, L_i admits a local framing $K_i \subseteq \mathbb{I}(E \oplus E'_1 \oplus E'_2)$ such that $[K_1, K_2] = 0$.

If (L, V_1, V_2) is a bicontact datum for (E, E'_1, E'_2), we denote $E_i = L_i \oplus \mathbb{L}_{V_i}(L_i)$ and $E''_i = E_i \oplus E'_i$ for $i = 1, 2$. We also write $E'' = E''_1 \oplus E''_2$.

Remark 77. If $E'' = \mathbb{T}M$, then the condition $K_i \subseteq \mathbb{I}(E \oplus E'_1 \oplus E'_2)$ is automatically satisfied. For instance, this happens if $L_1 \oplus W_1$ and $L_2 \oplus W_2$ define complementary transverse foliations of constant rank. In this case, the condition $[K_1, K_2] = 0$ is also satisfied by choosing local framings K_i that are pushed-forward from the corresponding leaves.

Example 78. Let $\eta_1, \eta_2 \in \Gamma(T^*M)$ be such that (η_1, η_2) is an ordinary bicontact structure i.e. such that there exist $k_1, k_2 \in \mathbb{Z}$ with the property that $\eta_1\eta_2(d\eta_1)^{k_1}(d\eta_2)^{k_2}$ is a volume form. Let $\xi_1, \xi_2 \in \Gamma(TM) \cap \ker(\mathbb{L}_{\eta_1}) \cap \ker(\mathbb{L}_{\eta_2})$ be such that $\langle \eta_i, \xi_j \rangle = \delta_{ij}$ and let $E_i' = \mathrm{span}(\eta_i, \xi_i)$ for $i = 1, 2$. If $E = (E_1' \oplus E_2')^\perp$, then $(TM \cap E, \mathrm{span}(\eta_1), \mathrm{span}(\eta_2))$ is a bicontact datum.

Remark 79. Let. $M = M_1 \times M_2$. If (L_i, V_i) are rank 1 contact data on M_i for $i = 1, 2$, then, arguing as in Section 7, $(L_1 \boxplus L_2, \pi_1^* V_1, \pi_2^* V_2)$ is a bicontact datum on M.

Example 80. Let $M_1 = M_2 = S^3$. If (L_i, V_i, W_i) are the rank 1 contact data described in Example 71, then $(L_1 \boxplus L_2, \pi_1^* V_1, \pi_2^* V_2)$ is a bicontact datum on $S^3 \times S^3$. Unless h^1 and h^2 both vanish (in which case we recover the standard bicontact structure on $S^3 \times S^3$ of [3]), this bicontact datum does not define a classical bicontact structure.

Lemma 81. *Let (L, V_1, V_2) be a bicontact datum for (E, E_1', E_2'). Then*

i) E_1 and E_2 are orthogonal split structures;
ii) (L_1, V_1) and (L_2, V_2) are rank 1 contact data;
iii) $V_i \oplus W_i \subseteq \mathbb{I}(E_1'') \cap \mathbb{I}(E_2'')$;
iv) L_i admits local framings $K_i \subseteq \mathbb{I}(E_1'') \cap \mathbb{I}(E_2'')$ for $i = 1, 2$.

Proof: Choose generators $\mathbf{e}_1 \in V_1$ and $\mathbf{e}_2 \in V_2$. By assumption $\mathbb{L}_{V_i}(L_i)$ has the same rank as L_i. Therefore, arguing as in the proof of Lemma 69, $\mathbb{L}_{V_i}(L_i)$ is maximal isotropic in E_i and thus $E_1, E_2 \in \mathbb{E}(M)$. Since for each $\mathbf{x}_i \in \Gamma(L_i)$

$$\langle \mathbf{x}_1, \mathbb{L}_{\mathbf{e}_2}\mathbf{x}_2 \rangle = \mathbb{L}_{\mathbf{e}_2}\langle \mathbf{x}_1, \mathbf{x}_2 \rangle - \langle \mathbb{L}_{\mathbf{e}_2}\mathbf{x}_1, \mathbf{x}_2 \rangle = 0$$

and similarly, using $[\mathbf{e}_1, \mathbf{e}_2] = 0$,

$$\langle \mathbb{L}_{\mathbf{e}_1}\mathbf{x}_1, \mathbb{L}_{\mathbf{e}_2}\mathbf{x}_2 \rangle = \mathbb{L}_{\mathbf{e}_2}\langle \mathbb{L}_{\mathbf{e}_1}\mathbf{x}_1, \mathbf{x}_2 \rangle - \langle \mathbb{L}_{\mathbf{e}_2}\mathbb{L}_{\mathbf{e}_1}\mathbf{x}_1, \mathbf{x}_2 \rangle = -\langle \mathbb{L}_{\mathbf{e}_1}\mathbb{L}_{\mathbf{e}_2}\mathbf{x}_1, \mathbf{x}_2 \rangle = 0$$

we conclude that E_1 and E_2 are orthogonal. In order to show that (L_i, V_i) is a contact datum, we only need to check condition 2) in Definition 66 since the remaining conditions are consequences of the assumption that $(L, V_1 \oplus V_2, W_1 \oplus W_2)$ is a rank 2 contact datum. Observe that for each $\mathbf{x}, \mathbf{y} \in \Gamma(L_1) \oplus W_1$

$$\mathbb{L}_{\mathbf{e}_2}[\mathbf{x}, \mathbf{y}] = [\mathbb{L}_{\mathbf{e}_2}\mathbf{x}, \mathbf{y}] + [\mathbf{x}, \mathbb{L}_{\mathbf{e}_2}\mathbf{y}] = 0$$

and

$$\langle [\mathbf{x}, \mathbf{y}], \mathbf{e}_2 \rangle = \mathbb{L}_{\mathbf{x}}\langle \mathbf{y}, \mathbf{e}_2 \rangle - \langle \mathbf{y}, \mathbb{L}_{\mathbf{x}}\mathbf{e}_2 \rangle = 0\,.$$

Since $\Gamma(L) \oplus W$ is closed under the Dorfman bracket and $\Gamma(L_1) \oplus W_1$ is the subspace of $\Gamma(L) \oplus W$ orthogonal to $\mathbf{e}_2$ and annihilated by $\mathbb{L}_{\mathbf{e}_2}$, we conclude that $\Gamma(L_1) \oplus W_1$ is also closed under the Dorfman bracket. Hence (L_1, V_1) is a rank 1 contact datum and, by the same token, so is (L_2, V_2). Since

$\mathbb{L}_{e_1}(K_2 \oplus \mathbb{L}_{\mathbf{e}_2}(K_2)) = 0$, iii) is proved if we show that W_1 normalizes E_2''. To see this, observe that $[W_1, K_2] \subseteq \Gamma(L) \cap \ker \mathbb{L}_{\mathbf{e}_1} = L_2$. This implies that $\mathbb{L}_{\mathbf{e}_2}[W_1, K_2] \subseteq E_2$ and thus $[W_1, \mathbb{L}_{\mathbf{e}_2} K_2]$ are sections of E_2. Since $[W_1, V_2 \oplus W_2] = 0$, this concludes the proof of iii). Let K_1 and K_2 be local framings as in Definition 76. Then $0 = \mathbb{L}_{\mathbf{e}_2}[K_1, K_2] = [K_1, \mathbb{L}_{\mathbf{e}_2} K_2]$ and similarly $[\mathbb{L}_{\mathbf{e}_1} K_1, K_2] = 0$ so that $0 = \mathbb{L}_{\mathbf{e}_2}[\mathbb{L}_{\mathbf{e}_1} K_1, K_2] = [\mathbb{L}_{\mathbf{e}_1} K_1, \mathbb{L}_{\mathbf{e}_2} K_2]$. We conclude that $K_i \oplus \mathbb{L}_{\mathbf{e}_i}(K_i)$ are mutually commuting local framings of E_1 and E_2. □

Definition 82. Let (L, V_1, V_2) be a bicontact datum for (E, E_1', E_2') and let $J \in \mathrm{CRF}(E'')$. We say that (J, L, V_1, V_2) is a Hermitian bicontact datum for (E, E_1', E_2') if

1) $V_1 \oplus W_1 \subseteq \mathbb{I}(J)$;
2) $J(V_1) = V_2$ and $J(W_1) = W_2$;
3) $J(\Gamma(L) \oplus \mathrm{span}(W_1 \oplus W_2)) \subseteq \Gamma(L) \oplus \mathrm{span}(W_1 \oplus W_2)$.

Example 83. Let (η_1, η_2) be a bicontact structure on M and let E, E_1', E_2' be as in Example 78. If in addition M is endowed with a Hermitian structure $(\mathcal{J}, g)$, then M is said [3] to be a *Hermitian bicontact manifold* provided that there exist $\xi_1, \xi_2 \in \Gamma(TM)$ infinitesimal automorphisms of the Hermitian structure such that $\mathcal{J}(\xi_1) = \xi_2$ provided that η_i is dual to ξ_i with respect to the metric g. As proved in [3], these assumptions imply that $(TM \cap E, \mathrm{span}(\eta_1), \mathrm{span}(\eta_2))$ is a bicontact datum. If $J = \mathcal{J} \oplus (-\mathcal{J}^*)$ then all conditions in Definition 82 are met, except possibly for $V_1 \subseteq \mathbb{I}(J)$ which is equivalent to the requirement that $d\eta_1$ is of bidegree $(1,1)$ with respect to $\mathcal{J}$.

Example 84. Let $(L_1 \boxplus L_2, \pi_1^* V_1, \pi_2^* V_2)$ be the bicontact datum on $S^3 \times S^3$ introduced in Example 80 and let $J_i \in \mathrm{SGF}(M_i)$ be as in Example 17. If Ψ is as in Example 56, then $(J_1 \boxplus_\Psi J_2, L_1 \boxplus L_2, \pi_1^* V_1, \pi_2^* V_2)$ is a Hermitian bicontact datum for $(E_1 \boxplus E_2, \pi_1^* E_1', \pi_2^* E_2')$.

Lemma 85. *If (J, L, V_1, V_2) is a Hermitian bicontact datum for (E, E_1', E_2'), then*

i) $J(E_1) \subseteq E_1$, $J(E_2) \subseteq E_2$ and $J(E_1' \oplus E_2') \subseteq E_1' \oplus E_2'$;
ii) If J_1 (resp. J_2) is the restriction of J to E_1 (resp. E_2) and Ψ denotes the restriction of J to $E_1' \oplus E_2'$, then $(J_1, J_2, V_1 \oplus W_1, V_2 \oplus W_2, \Psi)$ is a Morimoto datum.

Proof: Let $\mathbf{e}_1 \in V_1$, $\mathbf{e}_2 \in V_2$ be generators. Since $\mathbf{e}_1 \in \mathbb{I}(J)$ and $\mathbf{e}_2 = J(\mathbf{e}_1)$, then $\mathbf{e}_2 \in \mathbb{I}(J)$ by Lemma 24. In particular, $J(\ker(\mathbb{L}_{\mathbf{e}_2})) \subseteq \ker(\mathbb{L}_{\mathbf{e}_2})$. Moreover, $J(V_1) = V_2$ together with the orthogonality of J imply that $\mathbf{x} \in \Gamma(E'')$ is orthogonal to both V_1 and V_2 if and only if $J(\mathbf{x})$ is. Since by assumption $J(L_1) \subseteq L \oplus \mathrm{span}(W_1 \oplus W_2)$ and L_1 is the subbundle of $L \oplus$

$\mathrm{span}(W_1 \oplus W_2)$ orthogonal to both $V_1 \oplus V_2$ and annihilated by $\mathbb{L}_{\mathbf{e}_2}$, we conclude that $J(L_1) \subseteq L_1$. Since J commutes with $\mathbb{L}_{\mathbf{e}_1}$, this implies that $J(\mathbb{L}_{\mathbf{e}_1}(L_1) \subseteq \mathbb{L}_{\mathbf{e}_1}(L_1)$ and thus $J(E_1) \subseteq E_1$. Similarly, $J(E_2) \subseteq E_2$. From Lemma 81 we see that $V_i \oplus W_i \subseteq \mathbb{I}(E_1'') \cap \mathbb{I}(E_2'')$ and that L_i admits local framings $K_i \subseteq \mathbb{I}(E_1'') \cap \mathbb{I}(E_2'')$. This proves the lemma since $(V_1 \oplus W_1, V_2 \oplus W_2, \Psi)$ is by construction an admissible triple. □

Definition 86. We refer to $(J_1, J_2, V_1 \oplus W_1, V_2 \oplus W_2, \Psi)$ as in Lemma 85 as *the Morimoto datum corresponding to the Hermitian bicontact datum* (J, L, V_1, V_2).

Definition 87. A Hermitian bicontact datum (J, L, V_1, V_2) is adaptable if for $i = 1, 2$, L_i admits a local framing K_i such that

1) $K_1, K_2 \in \mathbb{I}(E'')$;
2) $[K_1, K_2] = 0$;
3) $d\langle J\mathbf{x}_i, \mathbb{L}_{V_i}(\mathbf{y}_i)\rangle \in \Gamma(E_i'')$ for any $\mathbf{x}_i, \mathbf{y}_i \in K_i$.

If K_i satisfies the above conditions, we say that K_i is *an adapted local framing of* L_i.

Example 88. ofLet M be a Hermitian bicontact manifold as in Example 83. Then $L_1 \oplus \mathrm{span}(\xi_1)$ and $L_2 \oplus \mathrm{span}(\xi_2)$ define transverse foliations which by Lemma 85 are preserved by $\mathcal{J}$. Therefore, J induces almost complex structures ϕ_i on the leaves S_i of $L_i \oplus \mathrm{span}(\xi_i)$. If for $i = 1, 2$ we let K_i be the push forward under the inclusion map of a local framing of TS_i, then K_i is an adapted local framing L_i. Therefore, if $d\eta_1$ is of bidegree $(1, 1)$ then the Hermitian bicontact datum constructed in Example 83 is adaptable.

Remark 89. Generalizing Example 84, let $(L_1 \boxplus L_2, \pi_1^* V_1, \pi_2^* V_2)$ be as in Remark 77 and let (J_i, L_i, V_i) be normal contact data for $i = 1, 2$. Given $\Psi \in \mathrm{CRF}(E_1' \boxplus E_2')$ such that $\Psi(V_1) \subseteq V_2$ and $\Psi(W_1) \subseteq \Psi(W_2)$, then $(J_1 \boxplus_\Psi J_2, L_1 \boxplus L_2, \pi_1^* V_1, \pi_2^* V_2)$ is an adaptable Hermitian bicontact datum.

Theorem 90 (Abstract Blair-Ludden-Yano Theorem)**.** *Let* (J, L, V_1, V_2) *be an adaptable Hermitian bicontact datum and let* $(J_1, J_2, V_1 \oplus W_1, V_2 \oplus W_2, \Psi)$ *be the corresponding Morimoto datum. Then* (J_1, L_1, V_1) *and* (J_2, L_2, V_2) *are normal contact data.*

Proof: Let K_1 and K_2 be adapted local framings of L_1 and L_2, respectively. Since L_i is maximal isotropic, combining Lemma 81 with Lemma 69, we see that $\mathbb{L}_{V_i}(L_i)$ is also maximal isotropic. Therefore, $K_i \oplus \mathbb{L}_{V_i}(K_i)$ is an adapted local framing of E_i''. Since by assumption $J \in \mathrm{CRF}(E'')$, then the Abstract Morimoto Theorem implies that $(J_1, V_1 \oplus W_1)$ and $(J_2, V_2 \oplus W_2)$ are normal pairs. As shown in the proof of Lemma 85, J_i preserves L_i and

thus $L_i \oplus \mathrm{span}(W_i)$. Therefore, (J_i, L_i, V_i) is a normal contact datum for (E_i, E_i').

Corollary 91 ([3]). *Let M be a Hermitian bicontact manifold with $d\eta_1$ of bidegree $(1,1)$. Then M is locally the product of two normal contact manifolds.*

Proof: Example 88 shows that the Hermitian bicontact datum of the Hermitian bicontact manifold M is adaptable and thus (J_1, L_1, V_1) and (J_2, L_2, V_2) are normal contact data by the Abstract Blair-Ludden-Yano Theorem. By Corollary 47, (J_i, L_i, V_i) induce normal contact data on the leaves S_i of $L_i \oplus \mathrm{span}(\xi_i)$. As observed in Example 74, this implies that each leaf inherits the structure of normal contact manifold. □

References

[1] Marco Aldi and Daniele Grandini, *Generalized contact geometry and T-duality*, J. Geom. Phys. **92** (2015), 78–93.

[2] Gianluca Bande and Amine Hadjar, *On normal contact pairs*, Internat. J. Math. **21** (2010), no. 6, 737–754.

[3] David E. Blair, Gerald D. Ludden, and Kentaro Yano, *Geometry of complex manifolds similar to the Calabi-Eckmann manifolds*, J. Differential Geometry **9** (1974), 263–274.

[4] Eugenio Calabi and Beno Eckmann, *A class of compact complex manifolds which are not algebraic*, Annals of Mathematics **58** (1953), 494–500.

[5] Edward G. Effros, *Transformation Groups and C^*-algebras*, Ann. of Math. (2) **81** (1965), no. 2, 38–55.

[6] Ralph R. Gomez and Janet Talvacchia, *On products of generalized geometries*, Geom. Dedicata **175** (2015), 211–218.

[7] ______, *Generalized CoKähler Geometry and an Application to Generalized Kähler Structures*, available at `arXiv:1502.07046`.

[8] Marco Gualtieri, *Generalized complex geometry*, Ann. of Math. (2) **174** (2011), no. 1, 75–123.

[9] Nigel Hitchin, *Lectures on generalized geometry*, Surveys in differential geometry. Volume XVI. Geometry ofspecial holonomy and related topics, Surv. Differ. Geom., vol. 16, Int. Press, Somerville, MA, 2011, pp. 79–124.

[10] David Iglesias Ponte and Aïssa Wade, *Integration of Dirac-Jacobi structures*, J. Phys. A **39** (2006), no. 16, 4181–4190.

[11] Anton Kapustin, *A-branes and noncommutative geometry* (2005), available at `hep-th/0502212`.

[12] A. Morimoto, *On Normal Complex Structures*, J. Math. Soc. Japan **15** (1963), 420–236.

[13] Hisao Nakagawa, *On framed f-manifolds*, Kōdai Math. Sem. Rep. **18** (1966), 293–306.

[14] ______, *f-structures induced on submanifolds in spaces, almost Hermitian or Kaehlerian*, Kōdai Math. Sem. Rep. **18** (1966), 161–183.

[15] Yat Sun Poon and Aïssa Wade, *Generalized contact structures*, J. Lond. Math. Soc. (2) **83** (2011), no. 2, 333–352.
[16] Ken'ich Sekiya, *Generalized almost contact structures and generalized Sasakian structures*, Osaka J. Math. **52** (2015), no. 1, 43–59.
[17] Izu Vaisman, *Generalized CRF-structures*, Geom. Dedicata **133** (2008), 129–154.